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LETTER TO THE EDITOR

Some properties of free tensor potentials

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Abstract. Free tensor potentials of arbitrary rank in covariant 'Gupta–Bleuler' and non-covariant 'Coulomb' gauges are shown to be unitarily equivalent as in the case of quantum electrodynamics. The corresponding Euclidean potentials can be constructed and some of their properties are discussed.

The recent revival of interest in massless field theory of higher spin in both flat and curved space–time (see for examples Fronsdal 1978a, b, Fang and Fronsdal 1978, Berends *et al* 1979, Curtright 1979) has motivated us to generalise our previous results on massless potentials, although in a different spirit. This short note can be considered as a supplement to our earlier papers (Lim 1979a, b), with the purpose of extending the results on free electromagnetic potentials and linearised gravitational potentials to the case of tensor potentials for massless particles with arbitrary integer spin.

Following Strocchi and Wightman (1974), a local and covariant gauge for a rank-s tensor potential $A^{\mu_1\cdots\mu_s}(x)$ is specified by $\{A^{\mu_1\cdots\mu_s}(x), \mathscr{H}, \langle\cdot\,,\cdot\rangle, \mathscr{H}'\}$ where $\langle\cdot\,,\cdot\rangle$ is an indefinite, non-degenerate, Hermitian sesquilinear form on a Hilbert space $\mathscr{H}$, $\mathscr{H}'\subset\mathscr{H}$ is a closed subspace on which $\langle\cdot\,,\cdot\rangle$ is semidefinite, and $A^{\mu_1\cdots\mu_s}(x)$ is an operator-valued distribution in $\mathscr{H}$, satisfying a set of Wightman-like axioms. The two-point function $W^{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}(x-y)$ of $A^{\mu_1\cdots\mu_s}(x)$ in a local covariant gauge (or Gupta–Bleuler gauge) is not positive semidefinite and it contains gauge parameters. Denote by H the completion of the tensor-valued Schwartz test-function space $\mathscr{S}(\mathbb{R}^4)\times\mathbb{C}^{4s}$ with respect to the sesquilinear form

$$\langle f, g\rangle = \sum_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}\iint \overline{f_{\mu_1\ldots\mu_s}(x)}\,W^{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}(x-y)g_{\nu_1\ldots\nu_s}\,\mathrm{d}^4x\,\mathrm{d}^4y.$$

Define a closed subspace

$$H' = \{f\in H|p\tilde{f} = p^{\mu_l}\tilde{f}_{\mu_1\ldots\mu_l\ldots\mu_s}(p) = 0 \quad \text{for any } \mu_l \text{ a.e. on } C_+\}$$

where $\tilde{f}$ is the Fourier transform of f and C_+ is the mantle of the forward lightcone. The form $\langle\cdot\,,\cdot\rangle$, when restricted to H', becomes positive semidefinite. The one-particle physical space H_G for the massless spin-s boson in covariant gauge is then given by the completion of the quotient space H'/H'', where H'' is the kernel of the restricted form.

Note that H' is independent of gauge parameters, hence the physical equivalence of all covariant Gupta–Bleuler gauges for $A^{\mu_1\cdots\mu_s}(x)$.

The tensor potential in the non-covariant 'Coulomb' gauge $A_C^{\mu_1\cdots\mu_s}(x)$ is specified by $A_C^{\mu_1\cdots\mu_s}(x)=0$ if any $\mu_l=0$ and

$$\sum_{i_l} A_C^{i_1\cdots i_l\cdots i_s}(x) = 0.$$

The corresponding one-particle space is given by the closed subspace

$$H_{\mathrm{C}} = \{f \in H' | f_{\mu_1 \dots \mu_s} = 0 \text{ if any } \mu_l = 0\}.$$

The sesquilinear form $\langle \cdot , \cdot \rangle$ is positive on H_{C}.

The physical equivalence of the tensor potentials in these two gauges can be shown by establishing a unitary map between H_{G} and H_{C}. Such a unitary map exists and is given by

$$\gamma : \tilde{f}_{\mu_1 \dots \mu_s} \to \tilde{f}_{\mu_1 \dots \mu_s} - \sum_{c(\mu_l)} \frac{p_{\mu_l} \tilde{f}_{\mu_1 \dots \mu_{l-1} 0 \mu_{l+1} \dots \mu_s}}{p_0} + \sum_{c(\mu_l \mu_r)} \frac{p_{\mu_l} p_{\mu_r} \tilde{f}_{\mu_1 \dots \mu_{l-1} 0 \mu_{l+1} \dots \mu_{r-1} 0 \mu_{r+1} \dots \mu_s}}{p_0^2}$$
$$- \dots + (-1)^{s-1} \sum_{c(\mu_1 \dots \mu_{s-1})} \frac{p_{\mu_1} \cdots p_{\mu_{s-1}} \tilde{f}_{000 \dots 0 \mu_s}}{p_0^{s-1}} + (-1)^s \frac{p_{\mu_1} \cdots p_{\mu_s} \tilde{f}_{000 \dots 0}}{p_0^s}$$

where $\Sigma_{c(\mu_1 \dots \mu_l)}$ denotes summation over all possible distinct combinations of indices $(\mu_1 \dots \mu_l)$. For example, for $s = 3$,

$$\gamma : \tilde{f}_{\mu_1 \mu_2 \mu_3} \to \tilde{f}_{\mu_1 \mu_2 \mu_3} - \left(\frac{p_{\mu_1} \tilde{f}_{0 \mu_2 \mu_3} + p_{\mu_2} \tilde{f}_{\mu_1 0 \mu_3} + p_{\mu_3} \tilde{f}_{\mu_1 \mu_2 0}}{p_0} \right)$$
$$+ \left(\frac{p_{\mu_1} p_{\mu_2} \tilde{f}_{00 \mu_3} + p_{\mu_1} p_{\mu_3} \tilde{f}_{0 \mu_2 0} + p_{\mu_2} p_{\mu_3} \tilde{f}_{\mu_1 00}}{p_0^2} \right) - \frac{p_{\mu_1} p_{\mu_2} p_{\mu_3} \tilde{f}_{000}}{p_0^3}.$$

The proof is similar to that for spin-1 and spin-2 cases. It can be shown by a straightforward calculation that $(\gamma \tilde{f})\mu_1 \dots \mu_s = 0$ if any $\mu_l = 0$. H'' is the kernel of γ since $\gamma h = 0$ implies $h \in H''$, and $H_{\mathrm{C}} = \gamma H'$. γ is well defined by restricting to the forward lightcone the Taylor expansion about $p = 0$. Hence γ defines a unitary equivalence $H_{\mathrm{C}} \simeq H_{\mathrm{G}} = H'/H''$.

The corresponding Euclidean tensor potential can be constructed as follows. The two-point Schwinger function $S_{i_1 \dots i_s, j_1 \dots j_s}$ is obtained by the following matrix transformation in addition to the usual analytic continuation to pure imaginary time:

$$S_{i_1 \dots i_s, j_1 \dots j_s}(x_{\mathrm{E}} - y_{\mathrm{E}}) = B^{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s}_{i_1 \dots i_s, j_1 \dots j_s} W_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s}(\bar{x} - \bar{y}, \mathrm{i}(x_0 - y_0)),$$

where

$$B^{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s}_{i_1 \dots i_s, j_1 \dots j_s} = \sum_s (B^{\mu_1}_{i_1} B^{\mu_2}_{i_2} + \tfrac{1}{4} \delta_{i_1 i_2} g^{\mu_1 \mu_2}) \dots (B^{\mu_{s-1}}_{i_{s-1}} B^{\mu_s}_{i_s} + \tfrac{1}{4} \delta_{i_{s-1} i_s} g^{\mu_{i_{s-1}} \mu_{i_s}})$$
$$\times (B^{\nu_1}_{j_1} B^{\nu_2}_{j_2} + \tfrac{1}{4} \delta_{j_1 j_2} g^{\nu_1 \nu_2}) \dots (B^{\nu_{s-1}}_{j_{s-1}} B^{\nu_s}_{j_s} + \tfrac{1}{4} \delta_{j_{s-1} j_s} g^{\nu_{s-1} \nu_s})$$

for even $s \geq 2$, and

$$B^{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s}_{i_1 \dots i_s, j_1 \dots j_s} = \sum_s (B^{\mu_1}_{i_1} B^{\mu_2}_{i_2} + \tfrac{1}{4} \delta_{i_1 i_2} g^{\mu_1 \mu_2}) \dots (B^{\mu_{s-2}}_{i_{s-2}} B^{\mu_{s-1}}_{i_{s-1}} + \tfrac{1}{4} \delta_{i_{s-2} i_{s-1}} g^{\mu_{s-2} \mu_{s-1}})$$
$$\times (B^{\nu_1}_{j_1} B^{\nu_2}_{j_2} + \tfrac{1}{4} \delta_{j_1 j_2} g^{\nu_1 \nu_2}) \dots (B^{\nu_{s-2}}_{j_{s-2}} B^{\nu_{s-1}}_{j_{s-1}} + \tfrac{1}{4} \delta_{j_{s-2} j_{s-1}} g^{\nu_{s-2} \nu_{s-1}})(B^{\mu_s}_{i_s} B^{\nu_s}_{j_s})$$

for odd $s \geq 1$, and Σ_s denotes the symmetrised sum over all distinct combinations of indices $(i_1 \mu_1, \dots, i_s \mu_s)$ and $(j_1 \nu_1, \dots, j_s \nu_s)$, and $B^{\mu_r}_{i_r} = 1$ if $i_r = \mu_r = 1, 2, 3$, $B^0_4 = i$, and $B^{\mu_1}_{i_1} = 0$ otherwise. This B-matrix is required to change all $g_{\mu\nu}$ to δ_{ij} in the two-point function. If we assume that the tensor potential is traceless, then the additional terms $\frac{1}{4} \delta_{ij} g^{\mu\nu}$ contribute only contact terms consisting of the delta function and its derivatives to the two-point Schwinger function. The resulting $S_{i_1 \dots i_s, j_1 \dots j_s}$ is traceless. In order for the two-point Schwinger function to be positive semidefinite, certain conditions need to be imposed on the gauge parameters in the two-point function. For example, in the case of electromagnetic potential in the covariant gauge, the Fourier transform of the two-point function is

$$\tilde{S}_{ij}(p_{\mathrm{E}}) = [\delta_{ij} + (F(p_{\mathrm{E}}^2) - 1)(p_i p_j / p_{\mathrm{E}}^2)](1/p_{\mathrm{E}}^2),$$

which is positive semidefinite only if the gauge parameter $F(p_{\mathrm{E}}^2)$ is a non-negative measurable function. We remark that there exists at least one such gauge, namely the analogue of the Feynman gauge in an electromagnetic potential, for which the two-point Schwinger function is positive semidefinite.

The one-particle space K of the Euclidean tensor potential in covariant gauge can be defined as the completion of the real tensor-valued Schwartz test function space $\mathscr{S}(\mathbb{R}^4) \times \mathbb{R}^{4s}$ with respect to the inner product

$$\langle f, g \rangle_K = \sum_{i_1 \ldots i_s, j_1 \ldots j_s} \iint f_{i_1 \ldots i_s}(x_{\mathrm{E}}) S_{i_1 \ldots i_s, j_1 \ldots j_s}(x_{\mathrm{E}} - y_{\mathrm{E}}) g_{j_1 \ldots j_s}(y_{\mathrm{E}}) \, \mathrm{d}^4 x_{\mathrm{E}} \, \mathrm{d}^4 y_{\mathrm{E}}.$$

The Euclidean tensor potential can be considered as the generalised Gaussian random tensor field over K with mean zero and covariance given by

$$E[\mathscr{A}(f)\mathscr{A}(g)] = \langle f, g \rangle_K.$$

In contrast to H in the Minkowski region, K has a positive metric.

The properties of the Euclidean tensor potential in covariant gauges is similar to the cases with $s \leqslant 2$. $\mathscr{A}$ does not satisfy the reflection property. Thus one cannot obtain a positive metric Hilbert space by the method of Osterwalder and Schrader (1973, 1975), which agrees with the result in relativistic theory. If additional conditions are imposed on the gauge parameters so that the Fourier transform of the Schwinger function $S(p_{\mathrm{E}})$ has an inverse which is a polynomial in p_{E}^2 and p_{i_l}, $l = 1, \ldots, s$, then the Euclidean tensor potentials in these covariant gauges are Markovian in the sense of Nelson (1973a, b). Again we remark that for arbitrary integer spin $s \geqslant 3$, there exists at least one gauge, the 'Feynman' gauge, for which the Euclidean potential is Markovian. In this case $S(p_{\mathrm{E}}) = C p_{\mathrm{E}}^{-2}$, where C is just a constant non-singular matrix so that $S^{-1}(p_{\mathrm{E}}) = p_{\mathrm{E}}^2 C^{-1}$.

For the noncovariant 'Coulomb' gauge the Euclidean potential satisfies the following conditions:

$$\mathscr{A}^{\mathrm{C}}_{i_1 \ldots i_s} = 0 \qquad \text{if any } i_l = 4,$$

and

$$\sum_i \partial_i \mathscr{A}^{\mathrm{C}}_{i, i_2 \ldots i_s}(x_{\mathrm{E}}) = 0.$$

The corresponding one-particle space is then given by the closed subspace

$$K_{\mathrm{C}} = K \cap \{f \in K \mid \sum_i f_{i, i_2 \ldots i_s} = 0 \quad \text{and } f_{i_1 \ldots i_s} = 0 \quad \text{if any } i_l = 4\}.$$

$\mathscr{A}^{\mathrm{C}}$ differs from the potential in a covariant gauge for it is reflexive but fails to satisfy Nelson's Markov property. However, it is Markovian with respect to special half-spaces or the Markov property of the second kind (Hegerfeldt 1974). A relativistic one-particle space with positive metric can be recovered from K^{C} by using the method of Osterwalder and Schrader. The proofs of these results follow closely those for spins

$s \leq 2$ (Lim 1979a, b). Finally we remark that in the Euclidean region, the closed subspace with zero divergence

$$K' = \{f \in K \mid \sum_i \partial_i f_{i,i_2 \ldots i_s} = 0\}$$

does not enter the theory. It is necessary if the relation between $\mathscr{A}^{\mathrm{C}}$ and $\mathscr{A}$ is considered. A result similar to the relativistic case exists for the Euclidean potentials, namely

$$K_{\mathrm{C}} \simeq K_{\mathrm{G}} \equiv K'/K''$$

where K'' is the subspace with vanishing norm, and K_{G} can be taken as the 'Euclidean Physical Space'.

Finally we remark that the above results may be generalised to a massless field curved space–time manifold in a non-trivial way. For example, the electromagnetic field in a smooth manifold can be considered as Ito's random current (Lim 1980, preprint in preparation).

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